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# Finite Groups in which the Nonnormal Subgroups Have Nontrivial Intersection

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A group $G$ is called a *Dedekind group* if any subgroup of $G$ is normal. All such groups were determined by Dedekind in 1897; they are the Abelian groups and the direct products of a quaternion group of order 8 with a periodic Abelian group having no element of order 4. If $G$ is not a Dedekind group we denote by $R(G)$ the intersection of all subgroups of $G$ which are not normal. Clearly a subgroup which is not normal contains a cyclic subgroup which is not normal; hence $R(G)$ is a cyclic characteristic subgroup of $G$. Our aim is the determination of all finite groups $G$ for which $R(G) \neq 1$.

First we give an example of such a group. We say that group $G$ of order a power of 2 is a *Q-group* if (1) $G$ has an Abelian subgroup $A$ of index 2, (2) $A$ is not elementary Abelian, and (3) $G$ is generated by $A$ and $x$, $x^{-1}ax = a^{-1}$ for all $a \in A$ and $x^2$ is an element of $A$ of order 2. For instance if $A$ is cyclic of order greater than 2, $G$ is a quaternion group. Again we obtain the non-Abelian Dedekind groups of order a power of 2 by taking $A$ to be the direct product of a cyclic group of order 4 with an elementary Abelian group and $x^2$ to be the unique element of $A$ which is a square. If $G$ is a $Q$-group but not a Dedekind group, then $R(G)$ is the group generated by $x^2$, for any nonnormal cyclic subgroup of $G$ is generated by $xa$ for some $a \in A$ and $(xa)^2 = x^2(x^{-1}ax)\, a = x^2$.

THEOREM 1. *Let $G$ be a group of order a power of a prime $p$. Suppose that $G$ is not a Dedekind group and that $R(G) \neq 1$. Then $p = 2$ and one of the following holds.*

(1) *$G$ is the direct product of a quaternion group of order 8, a cyclic group of order 4 and an elementary Abelian group.*

(2) *$G$ is the direct product of two quaternion groups of order 8 and an elementary Abelian group.*

(3) *$G$ is a $Q$-group.*

The proof of Theorem 1 is the main content of this paper. Using it the determination of the remaining groups $G$ for which $R(G) \neq 1$ is little more than an exercise on the basis theorem for Abelian groups.

THEOREM 2. *Suppose that $G$ is a finite group, that $G$ is not of prime-power order, that $G$ is not a Dedekind group and that $R(G) \neq 1$. Then one of the following holds.*

(a) *$G$ has an Abelian subgroup $A$ of exponent $kp^n$ where $n \geqslant 1$, $p$ is prime, and $(k, p) = 1$. $G/A$ is cyclic of order $p^r$ and if $Au$ generates $G/A$, $u$ can be so chosen that $u^{p^r}$ has order $p^n$. There exists an integer $\xi \equiv 1\ (p^n)$ such that $x^u = x^\xi$ for all $x \in A$.*

(b) *$G$ is the direct product of an Abelian group of odd order and one of the groups described in* (1) *or* (2) *of Theorem* 1.

(c) *$G$ has a subgroup $H$ of the kind described in* (a) *with $p = 2$ and $r = 1$. $H$ is of index 2 and if $G$ is generated by $H$ and $t$, $t$ can be so chosen that $u^t = u^{-1}$, $t^2 = u^{2^n}$, and $x^t = x^\eta$ for some $\eta \equiv -1(2^n)$.*

(d) *$G$ has an Abelian subgroup $A$ of index 2. $G$ is generated by $A$ and $t$ where $t^2$ is an element of $A$ of order 2. If $x$ is an element of $A$, $x^t = x^\zeta$ for some $\zeta \equiv -1\ (2^n)$.*

(e) *$G$ is the direct product of $H$, a quaternion group of order* 8, *and an elementary Abelian 2-group, where $H$ is of odd order and is of the kind described in* (a).

*Notation.* If $X$ is a subset of a group $G$, $\mathrm{gp}\,\{X\}$ denotes the subgroup generated by $X$. If $X$ is a subset and $A$ a subgroup of $G$, $C_A(X)$ is the centralizer of $X$ in $A$. If $a, b$ are elements of a group, $a^b = a[a, b] = b^{-1}ab$. $|G|$ denotes the order of the group $G$. $H \leqslant G$ means that $H$ is a subgroup of $G$; $H < G$ means that $H \leqslant G$ but $H \neq G$.

## *Proof of Theorem* 1

LEMMA 1. *Let $H$ be a subgroup of a group $G$.*

(a) *If $H$ is not a Dedekind group, $R(G) \leqslant R(H)$.*

(b) *If $H$ does not contain $R(G)$, then $H$ is a Dedekind group.*

(c) *If $R(G) \neq 1$, any element of $G$ of order 2 is contained in the center of $G$.*

This is trivial. For (a) we observe that a nonnormal subgroup of $H$ is a nonnormal subgroup of $G$ and hence contains $R(G)$. For (b), if $H \ngeqslant R(G)$, then certainly $R(H) \ngeqslant R(G)$ and the result follows from (a). To prove (c) suppose that $x$ is an element of order 2. If $x \in R(G)$, $x$ generates a normal subgroup since $R(G)$ is cyclic. If $x \notin R(G)$, then $\mathrm{gp}\{x\} \not\geq R(G)$, so $x$ generates a normal subgroup by definition of $R(G)$. Then (c) follows at once since a normal subgroup of order 2 always lies in the center.

We shall prove Theorem 1 by discussing a number of special cases first.

LEMMA 2. *Let $G$ be a group of order a power of the prime $p$. Suppose that all the proper subgroups of $G$ are Abelian, that $G$ is not a Dedekind group, and that $R(G) \neq 1$. Then $p = 2$ and $G$ is generated by $x$ and $y$ with defining relations*

$$y^4 = 1, \qquad [y, x] = y^2, \qquad x^4 = 1.$$

$G$ is noncyclic and so $G$ has more than one subgroup of index $p$. Two of these intersect in a subgroup $Z$ of index $p^2$; $Z$ is evidently the centre of $G$. If $G = \mathrm{gp}\{Z, s, t\}$, then $\mathrm{gp}\{s, t\}$ is non-Abelian, and so $G = \mathrm{gp}\{s, t\}$. Also $[s, t] \in Z$ and so the derived group $G'$ is generated by $[s, t]$. Since $s^p \in Z$, $s^p = (s^p)^t = (s^t)^p = (s[s, t])^p = s^p[s, t]^p$. Hence $[s, t]^p = 1$ and $G'$ is of order $p$. Since $G$ is not a Dedekind group, $G$ has a nonnormal cyclic subgroup $J$. Thus $J \not\geq G'$ and so $J \cap G' = 1$. Hence $R(G) \cap G' = 1$. Let $a$ be an element of $J$ of order $p$. If $\mathrm{gp}\{a\}$ is not normal, then $R(G) < \mathrm{gp}\{a\}$, so $R(G) = 1$. Suppose $\mathrm{gp}\{a\}$ is normal: then $a$ lies in the center of $G$ since $G$ is of prime-power order. Let $G' = \mathrm{gp}\{b\}$, $N = \mathrm{gp}\{ab\}$. Thus $N$ is contained in the center of $G$, and since $G' \leqslant N$, $G/N$ is non-Abelian. If $G/N$ has a nonnormal subgroup $H/N$, then $H \not\geq G'N$, so $b \notin H$ and $a \notin H$. Hence $H \cap J = 1$ and $R(G) = 1$. Otherwise $G/N$ is a non-Abelian Dedekind group with two generators. Thus $p = 2$ and $G/N$ is a quaternion group of order 8. Hence, since $J \cap N = 1$, $|\, J \,| = 4$. Thus $JG'$ is Abelian of order 8, and if $G = \mathrm{gp}\{y, JG'\}$, $J = \mathrm{gp}\{x\}$, then $[x, y] = b$. Thus $(xy)^2 = x^2y^2b = ay^2b = y^2ab$. Since $y^2$ lies in the center of $G$, $y^2 \in \mathrm{gp}\{a, b\}$. We may assume that $y^2 = 1$ or $y^2 = b$. In the former case $R(G) = 1$ by Lemma 1(c). $G = \mathrm{gp}\{x, y\}$ since $\mathrm{gp}\{x, y\}$ is non-Abelian.

It follows at once that if $G$ is a non-Abelian $p$-group and $R(G) \neq 1$, then $p = 2$. For suppose $G$ is a minimal counterexample. By Lemma 1(a) every proper subgroup of $G$ is a Dedekind group. Since $p$ is odd, every proper subgroup of $G$ is Abelian. By Lemma 2, $p = 2$, a contradiction.

Throughout the rest of this section then we shall consider groups of order a power of 2. We shall make much use of groups which are the direct product of a quaternion group of order 8, a cyclic group of order 4, and an elementary Abelian 2-group. Let us then denote the class of all such groups by $\mathscr{U}$. Of course if $G \in \mathscr{U}$, $|\, R(G) \,| = 2$.

LEMMA 3. *Suppose that $G$ is a 2-group and that $G$ has a maximal subgroup $H \in \mathscr{U}$. If $R(G) \neq 1$, then either $G \in \mathscr{U}$ or $G$ is the direct product of two quaternion groups of order 8 and an elementary Abelian group.*

Suppose that $H$ is the direct product of $Q$, $C$, and $E$, where $Q$ is a quaternion group of order 8, $C$ is cyclic of order 4, and $E$ is elementary Abelian. We write $Q = \mathrm{gp}\{x, y\}$ and $C = \mathrm{gp}\{c\}$. Thus $x^4 = y^4 = c^4 = 1$ and $x^2 = y^2 = [x, y]$.

By Lemma 1(a), $R(G) = R(H) = \mathrm{gp}\{x^2c^2\}$. Hence none of $C$, $\mathrm{gp}\{x\}$, $\mathrm{gp}\{y\}$ contains $R(G)$; thus $C$, $\mathrm{gp}\{x\}$, and $\mathrm{gp}\{y\}$ are all normal. Hence $Q$ is normal and the elements of $G$ induce automorphisms of $Q$ of the form $x \to x^{\pm 1}$, $y \to y^{\pm 1}$. At most 4 automorphisms are thus induced, and so $(G : Z) \leqslant 4$, where $Z$ is the centralizer of $Q$ in $G$. Since $Z \cap Q$ is the centre of $Q$, $(Q : Q \cap Z) = 4$. Hence $QZ = G$. Thus there exists an element $a$ of $Z$ not in $H$. We have $a^2 \in Z \cap H = \mathrm{gp}\{x^2, c, E\}$ and so $a^4 \in \mathrm{gp}\{c^2\}$. We may assume that every element of order 2 lies in $H$, since otherwise by Lemma 1(c) $G$ is the direct product of $H$ and a cyclic group of order 2, so that $G \in \mathscr{U}$.

Now $[xa, y] = x^2$ cannot then be a power of $xa$, for it is easy to verify that this would imply $a^2 = 1$. Hence $\mathrm{gp}\{xa\}$ is not normal. Therefore $\mathrm{gp}\{xa\} \geqslant R(G)$ and $x^2c^2 = (xa)^{2m}$ for some odd integer $m$. Hence $a^{4m} = 1$, $a^4 = 1$, and $a^2 = c^2$. Thus $\mathrm{gp}\{a\} \ngeqslant R(G)$ and so $\mathrm{gp}\{a\}$ is normal. Thus $[a, c]$ is a power of $a^2$. But $[a, c] \neq 1$ since otherwise $(ac^{-1})^2 = 1$ and this has been excluded above. Hence $[a, c] = a^2 = c^2$ and $\mathrm{gp}\{a, c\}$ is a quaternion group of order 8. $G$ is therefore the direct product of $Q$, $\mathrm{gp}\{a, c\}$, and $E$, as required.

Lemma 4. *Suppose that the group $G$ is of order a power of 2, $G$ is not a Dedekind group and $R(G) > 1$. If there is an element of order 4 in the center of $G$, $G \in \mathscr{U}$.*

This is proved by induction on $|G|$. Let $z$ be an element of order 4 in the center of $G$. First suppose that there is a non-Abelian subgroup $M$ of index 2 containing $z$. Then the center of $M$ has an element of order 4, so $M$ is not a Dedekind group. Hence by Lemma 1 (a), $R(M) > 1$. By the inductive hypothesis $M \in \mathscr{U}$. By Lemma 3 either $G \in \mathscr{U}$ or $G$ is the direct product of two quaternion groups of order 8 and an elementary Abelian group. This last case is however not possible, since the center of such a group is elementary Abelian.

We may therefore suppose that every maximal subgroup containing $z$ is Abelian. We prove that $G$ is generated by $z$ and a quaternion group of order 8. Let $A$ be a non-Abelian subgroup of $G$ of the smallest possible order; thus every proper subgroup of $A$ is Abelian. Since every proper subgroup of $G$ containing $z$ is Abelian, $\mathrm{gp}\{A, z\} = G$. If $A$ is not a Dedekind group, then $R(A) > 1$ by Lemma 1(a). Hence by Lemma 2, $A$ is generated by $x$ and $y$ with defining relations $x^4 = y^4 = 1$ and $[y, x] = y^2$; hence $R(G) = R(A) = \mathrm{gp}\{x^2\}$. But $[yz, x] = y^2 \notin \mathrm{gp}\{yz\}$, so $\mathrm{gp}\{yz\}$ is not normal and $x^2 = y^2z^2$. Hence gp $\{x^{-1}z, y\}$ is a quaternion group of order 8 which together with $z$ generates $G$. And if $A$ is a Dedekind group then $A$ is a quaternion group of order 8 since every proper subgroup of A is Abelian.

Suppose then that $G = \text{gp}\{Q, z\}$, $Q = \text{gp}\{a, b\}$ and

$$a^2 = b^2 = [a, b], \qquad a^4 = 1.$$

Since $za^{-1}$ does not lie in the center of $G$, the order of $za^{-1}$ is greater than 2, that is, $z^2 \neq a^2$. Hence $z^2$ does not lie in the center of $Q$. Hence $z^2$ does not lie in $Q$. Hence $G$ is the direct product of $Q$ and gp $\{z\}$, and $G \in \mathscr{U}$.

We now prove Theorem 1. Thus suppose that $G$ is of order a power of 2, that $G$ is not a Dedekind group, and that $R(G) \neq 1$. We prove first that $G$ has a cyclic normal subgroup of order 4. Suppose that this is not the case. Let $T$ be the set of elements of $G$ of order at most 2. By Lemma 1(c), $T$ is contained in the center of $G$. Let $N/T$ be a normal subgroup of $G/T$ of order 2 and let $N = \text{gp}\{T, x\}$. Then $x^2$ is an element of $T$ and $x^2 \neq 1$. By assumption gp $\{x\}$ is not normal, so $R(G) = \text{gp}\{x^2\}$. Let $U$ be a subgroup of $T$ such that $T$ is the direct product of $R(G)$ and $U$. We show that $T/U$ is the only subgroup of $G/U$ of order 2. For if $Uy$ is an element of $G/U$ of order 2 and $y \notin T$, then $y^2 \notin R(G)$, so gp $\{y\}$ is normal, contrary to assumption. Also since $U$ is contained in the center and $G$ is non-Abelian, $G/U$ is not cyclic. Hence $G/U$ is a quaternion group of order $2^n$ for some $n \geq 3$. If $n > 3$, $G/U$ has a cyclic normal subgroup $V/U$ of order 8; then $V$ is Abelian and the squares of the elements of $V$ form a cyclic normal subgroup of order at least 4. Thus $G/U$ is a quaternion group of order 8. If $vU$ is any element of $G/U$ of order 4, then again gp $\{v, U\}$ is normal and Abelian, so that $v^2$ generates a normal subgroup. Hence $v^4 = 1$ and since gp $\{v\}$ is not normal, $v^2 = x^2$. Hence $G = \text{gp}\{U, a, b\}$ where $a^4 = 1, a^2 = b^2 = (ab)^2$. Therefore $a^b = a^{-1}$ and gp $\{a\}$ is normal. Thus $G$ has a cyclic normal subgroup $N$ of order 4.

On account of Lemma 4 we may assume that the center of $G$ contains no element of order 4. Thus the centralizer $C$ of $N$ is a proper subgroup of $G$, in fact $(G : C) = 2$. Suppose that $C$ is non-Abelian. Since the center of $C$ contains an element of order 4, $C$ is not a Dedekind group. Thus by Lemma 1(a), $R(C) \neq 1$ and by Lemma 4, $C \in \mathscr{U}$. The required result now follows from Lemma 3. We therefore suppose now that $C$ is Abelian. If $x$ is any element not in $C$, then $x^2 \in C$, so the centralizer of $x^2$ contains gp $\{C, x\} = G$. Since the center of $G$ contains no element of order 4, $x^4 = 1$. If $x^2 = 1$, then by Lemma 1(c) $G$ is the direct product of $C$ and gp $\{x\}$. But then $G$ is Abelian, contrary to our assumption. Hence any element which does not lie in $C$ has order 4.

Suppose next that there is a cyclic normal subgroup gp $\{a\}$ not contained in $C$. If $N = \text{gp}\{b\}$, then $1 \neq [a, b] \in \text{gp}\{a\} \cap \text{gp}\{b\}$. Hence $[a, b] = a^2 = b^2$

and $Q = \mathrm{gp}\,\{a, b\}$ is a quaternion group of order 8. If $Z$ is the centralizer of $a$ in $C$, then $(C : Z) = 2$ and $Z$ is the center of $G$. Thus $Z$ is elementary Abelian. Since $(G : Z) = 4 \leqslant (Q : Q \cap Z) = (QZ : Z)$, we have $G = QZ$. Thus $G$ is a Dedekind group, contrary to hypothesis.

No element outside $C$, then, generates a normal subgroup. Hence the square of any such element is the element of $R(G)$ of order 2. Hence if $a \notin C$, $(ax)^2 = a^2$ for any element $x$ of $C$, and $a^{-1}xa = x^{-1}$. Thus $G$ is a $Q$-group.

*Proof of Theorem* 2

Assuming now that $G$ is a group which is neither of prime-power order nor a Dedekind group but that $R(G) > 1$, let $p$ be a prime divisor of $|R(G)|$. By definition of $R(G)$, any subgroup of order prime to $p$ is normal. The Sylow $q$-subgroups for $q \neq p$ therefore generate a normal $p$-complement $N$ of $G$. By Lemma 1 $N$ is a Dedekind group.

Since $G$ is not a Dedekind group, $G$ has a subgroup $H$ which is not normal. $H \cap N$ is a normal $p$-complement of $H$ and is normal in $G$. Thus $H$ has a Sylow $p$-subgroup $P$ which is not normal in $G$. If $S$ is a Sylow $p$-subgroup of $G$ containing $P$, either $S$ is a Dedekind group and is not normal in $G$, or $S$ is not a Dedekind group and $R(S) > 1$, by Lemma 1.

We now distinguish various cases according to the structure of $S$ and $N$.

1. Suppose $S$ and $N$ are both Abelian. Since $S$ is not normal, $N$ will contain elements which do not commute with every element of $S$. For any such element $x$ of $N$, we have $C_S(x) < S$. We shall consider such an element $x$ of prime-power order. Then the Sylow $p$-subgroup of the group of automorphisms of $\mathrm{gp}\,\{x\}$ is cyclic. Since $\mathrm{gp}\,\{x\}$ is normal, $S/C_S(x)$ is cyclic, say of order $p^{r(x)}$. Suppose $S = \mathrm{gp}\,\{C_S(x), a\}$ and that $a$ is of order $p^m$. We shall make considerable use of the fact that if $y$ is any element of $S$ not lying in $C_S(x)$, then $[y, x]$ is not a power of $y$, so $\mathrm{gp}\,\{y\}$ is not normal and $R(G) \leqslant \mathrm{gp}\,\{y\}$; indeed $R(G) \leqslant \mathrm{gp}\,\{y^p\}$ since $R(G)$ is normal. In particular we see by taking $y = a^{p^{r(x)-1}}$ that $R(G) \leqslant \mathrm{gp}\,\{a^{p^{r(x)}}\}$. Hence $m > r(x)$ and $a^{p^{m-1}}$ lies in $R(G)$.

We prove next that the exponent of $C_S(x)$ is $p^{m-r(x)}$. If this is not so, $C_S(x)$ contains an element $b$ of order $p^{m-r(x)+1}$, for $a^{p^{r(x)}}$ is an element of $C_S(x)$ of order $p^{m-r(x)}$. By the basis theorem for Abelian groups $\mathrm{gp}\,\{a, b\}$ is the direct product of $\mathrm{gp}\,\{a\}$ and $\mathrm{gp}\,\{c\}$ for some element $c = a^\lambda b^\mu$. If $c \notin C_S(x)$, then $R(G) \leqslant \mathrm{gp}\,\{c\}$ by the above remark and $a^{p^{m-1}} \in \mathrm{gp}\,\{c\}$, contrary to the properties of direct products. Hence $c \in C_S(x)$, so $a^\lambda \in C_S(x)$ and $\lambda \equiv 0(p^{r(x)})$. Also $\mu$ is prime to $p$, so $c^{p^{m-r(x)}} = b^{\mu p^{m-r(x)}} \neq 1$. But $a^{p^{r(x)-1}}c \notin C_S(x)$, so $R(G) \leqslant \mathrm{gp}\,\{a^{p^{r(x)}}c^p\}$, from above. Hence $a^{p^{m-1}} = a^{\nu p^{r(x)}}c^{p\nu}$ for some $\nu$, and by the properties of direct products, $c^{p\nu} = 1$. Hence

$\nu \equiv 0(p^{m-r(x)})$, whence $a^{p^{m-1}} = 1$, a contradiction. The assertion is therefore proved. Thus the exponent of $S$ is $p^m$, and $m$ is independent of $x$ and $a$.

For $i = 0, 1, ..., m$, denote by $E_i$ the set of elements of $S$ of order at most $p^i$. We prove next that $C_S(x) = E_{m-r(x)}$. We have just shown that $C_S(x) \leqslant E_{m-r(x)}$. Now $S$ is the direct product of gp $\{a\}$ and some group $S_1$, since the order of $a$ is the exponent of $S$. Thus $S_1 \cap \text{gp}\,\{a\} = 1$, so $a^{p^{m-1}} \notin S_1$, $R(G) \leqslant S_1$, $S_1$ is normal in $G$ and $[S_1, N] = 1$. Thus $S_1 \leqslant C_S(x) \leqslant E_{m-r(x)}$, so $E_{m-r(x)} = \text{gp}\,\{S_1, a^{p^{m-r(x)}}\} \leqslant C_S(x)$, yielding the required result.

Now choose an element $y$ of $N$ of prime-power order so that $r = r(y)$ is as large as possible, and let $E = E_{m-r}$. Thus $E \leqslant E_{m-r(x)}$ for all relevant $x$, and so $E \leqslant C_S(N)$. But $C_S(N) \leqslant C_S(y) = E$, so $E = C_S(N)$. Also $S = \text{gp}\,\{E, u\}$, where $u$ is of order $p^m$ and $uE$ is of order $p^r$. Let $A$ denote the Abelian group $EN$. For each Sylow subgroup $Q$ of $A$ choose an element $z$ of maximal order, and write $z^u = z^\xi$. If $t \in Q$ and gp $\{t\} \cap \text{gp}\,\{z\} = 1$, then $t^u = t^\xi$ as is seen from the fact that $t^u$, $(tz)^u$ are powers of $t$, $tz$ respectively. Hence we can write

$$x^u = x^{\xi_Q}$$

for all $x \in Q$. If $\xi$ is an integer such that $\xi \equiv \xi_Q$ $(|Q|)$ for all $Q$, then in particular $\xi \equiv 1(p^{m-r})$ since $S$ is Abelian. $G$ thus has the structure described in Theorem 2(a) and $R(G) \geqslant \text{gp}\,\{a^{p^{m-1}}\}$.

2. Suppose $S$ is non-Abelian. Since either $S$ is a Dedekind group or $R(S) > 1$, it follows from Theorem 1 that $p = 2$, and $S$ has one of the three possible structures described in Theorem 1. In the first two cases $S$ is generated by certain subgroups, none of which contains $R(S) = R(G)$. These subgroups are therefore all normal, $S$ is normal, and $G$ has the structure described in Theorem 2(b). We assume then that $S$ is a $Q$-group (possibly a Dedekind group). Then $S$ has an Abelian subgroup $C$ of index 2 and by Lemma 1 either $CN$ is Abelian or $R(CN) > 1$. Of course $N$, being a Dedekind group of odd order, is Abelian.

Suppose first that $R(CN) > 1$. Then $H = CN$ is a group of the kind described in the first case, that is, in Theorem 2(a) with $p = 2$. Thus if $A$ is the centralizer of $N$ in $H$, $H$, $N$, and $A$ are normal in $G$; also there is an element $u$ of $S$ of order $2^m$ such that $H = \text{gp}\,\{u, A\}$ and $x^u = x^\xi$ for all $x \in A$ and a fixed integer $\xi$. If $S = \text{gp}\,\{C, t\}$, then $u^t = u^{-1}$ since $S$ is a $Q$-group. If $S$ is a Dedekind group, then $m = 2$ and $t^2 = u^2$. The fact that $t^2 = u^{2^{m-1}}$ in general follows from Lemma 1 and the fact that when $S$ is not a Dedekind group, $R(S) = \text{gp}\,\{t^2\}$ and $R(CN) \geqslant \text{gp}\,\{u^{2^{m-1}}\}$. Transforming the equation $x^u = x^\xi$ by $t$ shows that $u^2 \in C_H(N) = A$; hence $r = 1$. Every cyclic subgroup of $A$ is normal, and so there is an $\eta \equiv -1(2^m)$ such that $x^t = x^\eta$ for all $x \in A$. Thus $G$ is as described in Theorem 2(c).

If $CN$ is Abelian, $G$ is clearly as described in Theorem 2(d); the automorphism of $N$ induced by $t$ is seen from the first case applied to gp $\{t, N\}$.

3. In the remaining case $S$ is Abelian and $N$ is non-Abelian. Since $N$ is a Dedekind group, $|N|$ is even and $p$ is odd. The Sylow 2-subgroup $T$ of $G$ is thus a normal non-Abelian Dedekind group. If $U$ is a 2-complement containing $S$, then $U$ centralizes $T$. For if $x \in T$, gp $\{x\}$ is normal since gp $\{x\} \not\geq R(G)$. Also the automorphism group of a cyclic 2-group is a 2-group. Hence the only automorphism which $U$ induces on gp $\{x\}$ is the identity, and the assertion is proved. $G$ is thus the direct product of $U$ and $T$. Since $G$ is not a Dedekind group, neither is $U$; hence $R(U) > 1$ and $U$ is of the kind described in (a). Theorem 2 is thus completely proved.